Vibrations of a Pendulum with Oscillating Support and Extra Torque

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The motion of a driven planar pendulum with vertically periodically oscillating point of suspension and under the action of an additional constant torque is investigated. We study the influence of the torque strength on the transition to chaotic motions of the pendulum using Melnikov's analysis.

1 Introduction

The periodically driven planar pendulum is one of the simplest mechanical systems which can exhibit chaotic behaviour. It is usually modelled as a mathematical pendulum with linear damping and external or parametric excitation [1, 2]. Since the dynamics of various systems of various kind in mechanical [1, 2] or electrical [3, 4] engineering can be described by the same or the same type of differential equations as the one for the driven pendulum it is not a great surprise, that many methods invented to calculate and control chaotic vibrations of nonlinear systems have been tested on this system where the mathematical parameters appearing in the above mentioned equations can be given a well interpretable physical meaning.

In this note we focus on the planar pendulum with a vertically periodically moving suspension point with an additional constant torque [5, 6]. The corresponding differential equation has the following form:

$$ml^2\ddot{\phi} + k\dot{\phi} + m(lg - l\ddot{x})\sin\phi = T, \qquad x = a\sin\Omega t, \tag{1}$$

where m and l denote the point mass and the length of the mathematical pendulum and T is the extra constant torque applied at the suspension point. a is the amplitude of the vertical excitation $x(t)$ of the suspension point and Ω the corresponding frequency of excitation.

Introducing dimensionless time $\tau = \omega t$, Eq. 1 in dimensionless form is

$$\ddot{\phi} + \alpha\dot{\phi} + (1 + \gamma\Omega'^2\cos\Omega'\tau)\sin\phi = t_0, \tag{2}$$

where the corresponding constants are given by: $\omega = \sqrt{g/l}$, $\Omega' = \Omega/\omega$, $\alpha = k/(ml^2\omega)$, $\gamma = a/l$, $t_0 = T_0/(ml^2\omega^2)$.

Our goal is to investigate the dynamics of Eq. 2 concerning the occurrence of chaotic vibrations using different approximations.

2 Melnikov's analysis

In the first approximation, we assume that damping, excitation and extra torque are all small terms which we scale by introducing a small parameter ϵ to Eq. 2

$$\ddot{\phi} + \epsilon\tilde{\alpha}\dot{\phi} + (1 + \epsilon\tilde{\gamma}\Omega'^2\cos\Omega'\tau)\sin\phi = \epsilon\tilde{t}_0, \tag{3}$$

where $\tilde{\alpha}\epsilon = \alpha$, $\tilde{\gamma}\epsilon = \gamma$, $\tilde{t}_0\epsilon = t_0$, respectively. This allows us to apply Melnikov's method [7, 8] for the unperturbed symmetric potential [2]. The critical value of the excitation amplitude γ_c which is given by a simple zero of Melnikov function [7, 8]:

$$\gamma_c = \frac{4}{\pi\Omega'^4}\left|-\alpha + \frac{t_0\pi}{8}\right|\sinh\left(\Omega'\frac{\pi}{2}\right). \tag{4}$$

is (for $t_0 = 0$) very similar to that obtained in [2], with γ'_c, differing by a constant coefficient depending on the torque t_0:

$$\gamma_c = \gamma'_c\frac{\alpha}{\left|-\alpha + \frac{t_0\pi}{8}\right|}. \tag{5}$$

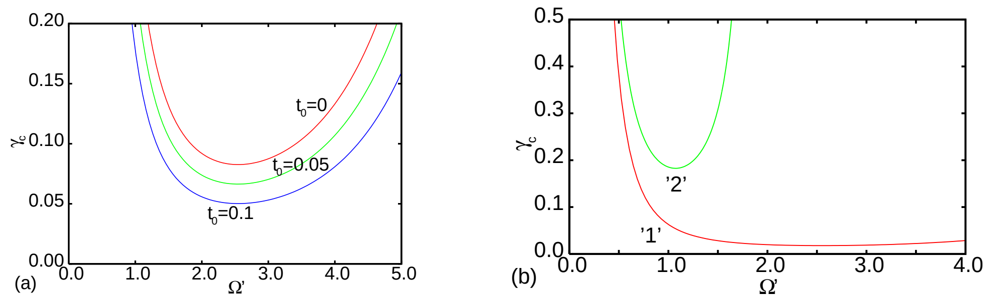

Fig. 1 (a) Dependence of critical value of the excitation amplitude γ_c versus Ω' for $\alpha = 0.1$ and three values of a constant torque $t_0 = 0.0$, 0.05 and 0.1 in the case of heteroclinic transition. (b) the same dependences for $t_0 = 0.2$ using two different approximations with heteroclinic - '1' and homoclinic - '2' transitions. Below the curves the system is regular while above transit to chaotic oscillations occurs. Note, different scales in Figs. 1a and b.

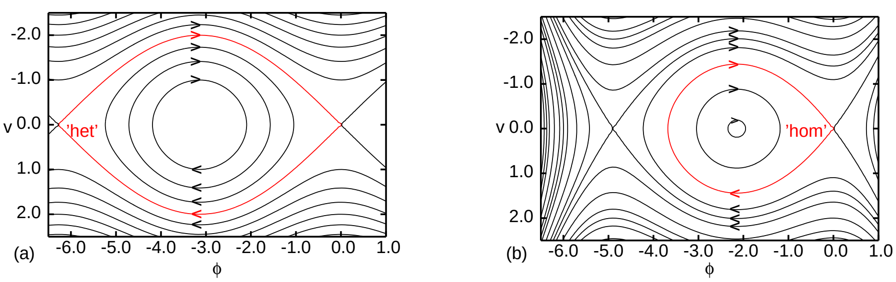

Fig. 2 Phase trajectories (where $v = \dot{\phi}$) for $t_0 = 0.2$ and two different approximations, with heteroclinic (a) and homoclinic (b) orbits (marked by 'het' and 'hom'), respectively.

In Fig. 1a γ_c versus Ω' is plotted, showing the influence of the torque on the existence of chaotic dynamics for (Eq. 3). Note that these results are obtained for small torque compared to damping $t_0/\pi 8 \ll \alpha$.

In the second approximation we treat the case of larger torque that is $t_0/(\pi 8) \sim \alpha$. Here we use the recently developed method for non-symmetric potentials [9]. This approach deals with the equation

$$\ddot{\phi} + \epsilon\tilde{\alpha}\dot{\phi} + (1 + \epsilon\tilde{\gamma}\Omega'^2 \cos\Omega'\tau)\sin\phi - t_0 \cong \ddot{\phi} + \epsilon\tilde{\alpha}\dot{\phi} + \left[1 - t_0 + \phi - \phi^3/6\right] + \epsilon\tilde{\gamma}\Omega'^2 \cos\Omega'\tau \sin\phi = 0, \quad (6)$$

where contrary to the first approximation t_0 is not taken to perturbations terms (contrary to Eq. 3) but included into the effective unperturbed potential. Consequently it breakes the potential symmetry and instead of a heteroclinc transition with two saddle points, a homoclinic transition with a single saddle point is obtained [5] is obtained (see Fig. 2).

In Fig. 1b we compare results of both approximations. Curve '1' corresponds to the small torque approach with heteroclinc transition while '2' represents the results with t_0 included in the effective potential (a homoclinic transition). Note, for small frequencies both curves match quite well but for higher frequencies there is a growing discrepancy between the two approximations.

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